Enumeration of No Strategy Games

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Definitions

Combinatorial game

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Game of No Strategy

A **game of no strategy** is a combinatorial game that has a predetermined winner based on the order of play, i.e. who plays first, who plays second, etc.

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$$C_n = \frac{1}{2} \sum_{i=1}^{n-1} \binom{n}{i} \binom{n-2}{i-1} C_i C_{n-i}$$

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$$C_n = \prod_{i=2}^{n} \binom{i}{2} = \frac{n!(n-1)!}{2^{n-1}}$$

Graphs

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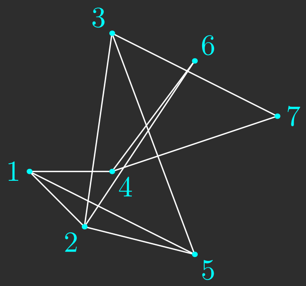

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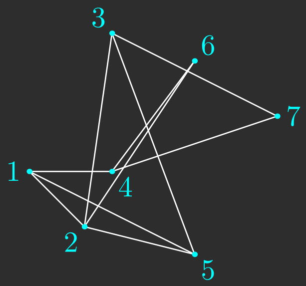

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Directed Graph

A **directed graph** is a graph whose edges are given a direction.

Game-Graph Connection

The Graph of a Game

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- Vertices: positions of game
- Edges: directed edge between vertices connected by a move

Isomorphic Games

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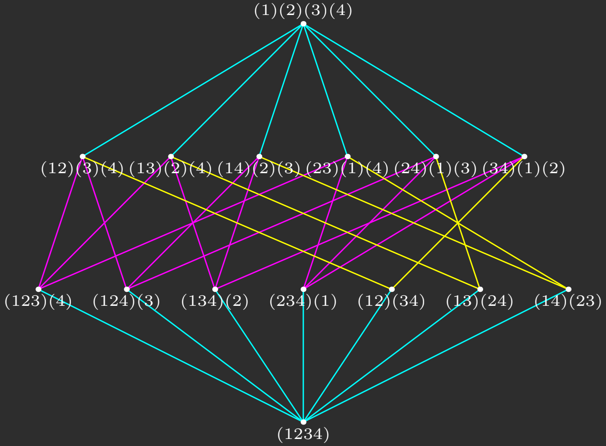

The graph of both games for $n = 4$

Isomorphic Games

Theorem

Suppose that the number of different ways to play the game is C_n. Then

$$C_n = \frac{1}{2}\sum_{i=1}^{n-1}\binom{n}{i}\binom{n-2}{i-1}C_i C_{n-i} = \prod_{i=2}^{n}\binom{i}{2} = \frac{n!(n-1)!}{2^{n-1}}.$$

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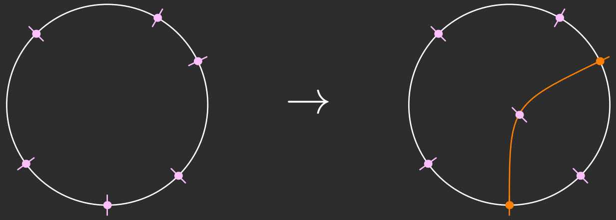

Proof of No Strategy

Euler characteristic

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- $m = n - 1$

Induction

Each move breaks the game into two smaller games with one less total vertex. By induction the game will take $n - 1$ moves.

Number of Ways to Play

Theorem

The number of ways to play x_n satisfies

$$x_n = \frac{n}{2} \sum_{i=1}^{n-1} \binom{n-2}{i-1} x_i x_{n-i}.$$

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Connection with Spanning Trees

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Connection with Spanning Trees

- number of ways to play equals the number of spanning trees on n vertices.
- $x_n = n^{n-2}$ (Cayley's formula)

Mozes's Game of Numbers

IMO 1986 #3

- Start: A regular pentagon (or, in general, any regular polygon) with integers $x_1, x_2, \ldots, x_n$ assigned to each vertex. The sum of the integers must be positive.

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- End: A polygon with all nonnegative vertices

Strictly Decreasing Monovariants

- Indices are taken mod n.
- Squares of differences:

$$f_1(x_1, x_2, x_3, x_4, x_5) = \sum_{i=1}^{5} (x_i - x_{i+2})^2$$

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- Arc sums:

$$f_3(x_1, x_2, x_3, \ldots, x_n) = \sum_{i=1}^{n} \sum_{j=i+1}^{n+i-2} |x_i + x_{i+1} + \cdots + x_{i+j}|$$

Fixed Length Game

Theorem

If the initial sum of the numbers S is positive, then the game will terminate in a fixed number of moves.

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Number of Moves (Alon et al.)

The game takes

$$\sum_{a_\lambda < 0} \frac{|a_\lambda|}{S}$$

moves, where the sum ranges over all negative arc sums.

Number of Ways to Play

Theorem

Beginning with numbers $-a, 2k+1, -2k+a, 0, 0, 0, \ldots$ *on an* $(m+2)$*-gon with* $k, m, a > 0$*, there are* $\binom{2mk}{ma}$ *ways to play.*

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- Corollary: Beginning with numbers $-k, 2k+1, -k, 0, 0, 0, \ldots,$ there are $\binom{2mk}{mk}$ ways to play.

A Chocolate Break

Statement of the Game

Consider a gridded $m \times n$ chocolate bar. The first player breaks the bar along one of its grid lines. Each move after that consists of taking any piece of chocolate and breaking it along existing grid lines, until only individual squares remain. The first player unable to break a piece loses.

A bar of chocolate

A Chocolate Break

Proof of No Strategy

Each move increases the number of chocolate pieces by 1. Since the game ends with mn individual squares, the $(mn-1)$th break must be the last.

Number of Ways to Play ($2 \times n$)

Theorem

Suppose the number of ways to play on a $2 \times n$ bar is B_n. Then

$$B_n = (2n-2)! + \sum_{m=1}^{n-1} \binom{2n-2}{2m-1} B_m B_{n-m}.$$

Number of Ways to Play ($m \times n$)

Theorem

Suppose the number of ways to play on a $m \times n$ bar is $A_{m,n}$. Then

$$A_{m,n} = \sum_{i=1}^{m-1} \binom{mn-2}{in-1} A_{i,n} A_{m-i,n} + \sum_{i=1}^{n-1} \binom{mn-2}{im-1} A_{m,i} A_{m,n-i}.$$

Values of $A_{2,n} = B_n$

n	B_n	factorization of B_n
1	1	1
2	4	2^2
3	56	$2^3 \cdot 7$
4	1712	$2^4 \cdot 107$
5	92800	$2^7 \cdot 5^2 \cdot 29$
6	7918592	$2^{10} \cdot 11 \cdot 19 \cdot 37$
7	984237056	$2^{10} \cdot 11 \cdot 59 \cdot 1481$
8	168662855680	$2^{12} \cdot 5 \cdot 11 \cdot 31 \cdot 24151$
9	38238313152512	$2^{15} \cdot 11 \cdot 571 \cdot 185789$
10	11106033743298560	$2^{17} \cdot 5 \cdot 11 \cdot 1607 \cdot 958673$
11	4026844843819663360	$2^{18} \cdot 5 \cdot 11 \cdot 97 \cdot 9371 \cdot 307259$
12	1784377436257886142464	$2^{19} \cdot 11^2 \cdot 569 \cdot 185833 \cdot 266009$

Values and factorizations of $A_{2,n} = B_n$

Values of $A_{m,n}$

	1	2	3	4	5
1	1	1	2	6	24
2	1	4	56	1712	92800
3	2	56	9408	4948992	6085088256
4	6	1712	4948992	63352393728	2472100837326848
5	24	92800	6085088256	2472100837326848	3947339798331748515840

Values of $A_{m,n}$

Properties of $A_{m,n}$

Theorem

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- Corollary: $\nu_2(B_n) \geq n$ for $n > 1$

Properties of B_n

Theorem

If $n \equiv 0$ *or* $1 \pmod 3$, *then* $B_n \equiv 2 \pmod 3$. *If* $n \equiv 2 \pmod 3$, *then* $B_n \equiv 1 \pmod 3$.

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Theorem

Given any positive integer n, *for all positive integers* k *dividing* B_i *for all* $\left\lfloor \frac{n+1}{2} \right\rfloor \leq i \leq n-1$ *and satisfying* $k | (2n-2)!$, *then* $k | B_j$ *for all* $j \geq n$.

Properties of B_n

Theorem

If $n \equiv 0$ or $1 \pmod 3$, then $B_n \equiv 2 \pmod 3$. If $n \equiv 2 \pmod 3$, then $B_n \equiv 1 \pmod 3$.

Theorem

Given any positive integer n, for all positive integers k dividing B_i for all $\lfloor \frac{n+1}{2} \rfloor \leq i \leq n-1$ and satisfying $k|(2n-2)!$, then $k|B_j$ for all $j \geq n$.

- Corollary 1: For all $i \geq 6$, $11|B_i$.
- Corollary 2: For all $i \geq 13$, $5|B_i$.

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- End: One remaining number — note that the final number is fixed

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- We assume zeroes and ones are indistinguishable, and that two moves are indistinguishable if they produce the same result.

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Define $f(m, n)$ to be the number of distinct ways to play the game with m zeroes and n ones.

Recursion

$$f(m, n) = f(m-1, n) + f(m+1, n-2) \quad \text{for} \quad m, n > 2.$$

Connection with Catalan Numbers

Definition

Let C_n be the number of strings of n X's and n Y's such that no segment of the string starting from the beginning has more X's than Y's. Then

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

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Theorem

$$f(0, 2n) = f(0, 2n+1) = C_n.$$

Connection with the Catalan Triangle

Definition

Let $C_{n,k}$ be the number of strings of n X's and k Y's such that no segment of the string starting from the beginning has more X's than Y's. Then

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$$f(n, 2k) = f(n, 2k+1) = C_{n,k}.$$

Future Research

- Sequence Bounds

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- Periodicity Rules

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- Games on 3D Surfaces

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- Mozes's Game of Numbers

Acknowledgements

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